

Lecture I: Parametric Oscillators

Key Idea and Examples:

- Oscillator with sinusoidally varying frequency
- Mathieu Equation

Contents:

- Ideas, trends
- Proof of nature of solutions
- Variation of parameters calculation for growth rate threshold

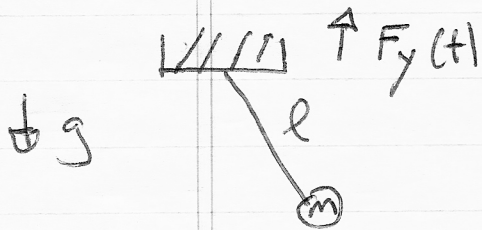
Result:

- Parametric growth calculation

2) Parametric Instability

→ consider pendulum with support acted on by vertical force

↓ + → down



so $g \Rightarrow g - F_y(t)/m$

$$\therefore \ddot{\theta} = \ddot{\theta} + \frac{g}{l} \theta \rightarrow \ddot{\theta} + \left(\frac{g}{l} - \frac{a(t)}{l} \right) \theta = 0$$

let $a(t) = a_0 \cos(\alpha t)$

see small ϕ limit, No. 12
 $\alpha \gg \omega_0$

$$\Rightarrow \ddot{\theta} + \omega_0^2 \theta - \frac{a_0 \cos(\alpha t)}{l} \theta = 0$$

of Mathieu's equation genre, i.e.

$$\left[\ddot{x} + \omega_0^2 (1 + a \cos(\gamma t)) x = 0 \right]$$

$\omega^2 = \omega^2(t)$, hence parametric oscillator

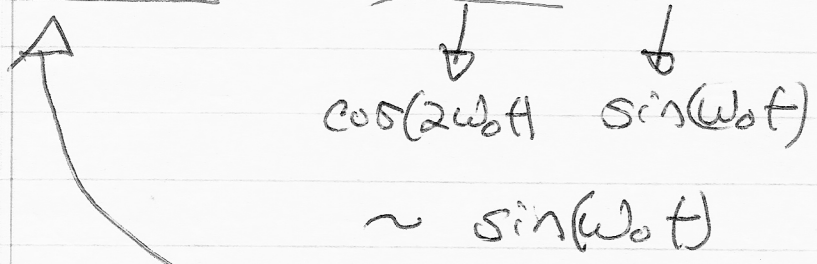
Parametric oscillator $\leftrightarrow \omega^2(t)$ periodic oscillation of effective frequency.

→ Some observations:

a) informal - consider what might happen?

for instability, observe can produce secularly if $\gamma \sim 2\omega_0$ via beat at fundamental

$$\ddot{x} + \omega_0^2 x + a \cos(\gamma t) \omega_0^2 x = 0$$



resonant drive of fundamental oscillator \Rightarrow secularly \Rightarrow instability (why?)

∴ Solution of oscillator at ω_0 beats with parameter oscillation \Rightarrow secularly

∴ parametric resonance at/near $\gamma \sim 2\omega_0$ (twice fundamental)

Note: here $\omega^2 = \omega^2(t) \Rightarrow \partial L / \partial t \neq 0$ energy not conserved

\Rightarrow work done on system (e.g. LGM oscillating pendulum support)

\Rightarrow source of energy for instability

! What is relation of this to 3-mode parametric instability calculation (2004) !

b) Formal (Floquet theory) $\left\{ \begin{array}{l} \text{what} \\ \text{Mathematics} \\ \text{Predicts} \end{array} \right.$
 \Rightarrow (What type solution possible)
 - $\omega(t)$ periodic, with period $T = 2\pi/\gamma$

$$\therefore \begin{cases} \omega(t+T) = \omega(t) \\ \text{eqn. invariant under } t \rightarrow t+T \end{cases}$$

\therefore if $x_1(t), x_2(t)$ are 2 independent solutions of basic eqn.

$\Rightarrow x_1(t), x_2(t)$ must transform to linear combinations of themselves upon $t \rightarrow t+T$ (linear eqn.)

and

can choose x_1, x_2 s/t

$$\begin{cases} x_1(t+T) = \mu_1 x_1(t) \\ x_2(t+T) = \mu_2 x_2(t) \end{cases}$$

(here "can choose" means can diagonalize transformation matrix)

\rightarrow most general functions having this property are:

$$\begin{cases} x_1(t) = \mu_1^{t/T} \pi_1(t) \\ x_2(t) = \mu_2^{t/T} \pi_2(t) \end{cases} \quad \left\{ \begin{array}{l} \text{where:} \\ \pi_i(t+T) = \pi_i(t) \end{array} \right.$$

- second, observe since linear equation
⇒ Wronskian constant

$$\dot{x}_2 x_1 - \dot{x}_1 x_2 = \text{const.}$$

$$\begin{matrix} x_2 & (\ddot{x}_1 + \omega^2(t) x_1) = 0 \\ x_1 & (\ddot{x}_2 + \omega^2(t) x_2) = 0 \end{matrix} \Rightarrow \frac{d}{dt} (x_2 \dot{x}_1 - \dot{x}_2 x_1) = 0$$

but

$$W(x_1, x_2) = (M_1, M_2)^{-1} W(x_1(t+T), x_2(t+T))$$

c.e. consider time translation by T

→ $M_1, M_2 = 1$ | $W(x_1, x_2) = \begin{pmatrix} M_1 & t/T \\ \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} M_1 & t/T \\ \pi_1 & \pi_2 \end{pmatrix} - \begin{pmatrix} M_1 & t/T \\ \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} M_2 & t/T \\ \pi_2 & \pi_1 \end{pmatrix}$

- Can also observe: $\begin{pmatrix} e^{(\ln M_2) t/T} e^{(\ln M_1) t/T \pi_1} \\ - e^{(\ln M_1) t/T \pi_1} e^{(\ln M_2) t/T \pi_2} \end{pmatrix}$

1) wells in oscillator reg, so
x(t) an integral → x* a solution

⇒

2) M_1, M_2 same as M_1^*, M_2^*
c.e.

$$\begin{matrix} M_1 = M_2^* \\ M_2 = M_1^* \end{matrix} \quad \underline{\text{or}} \quad \begin{matrix} M_1 = M_1^* \\ M_2 = M_2^* \end{matrix} \quad \left. \vphantom{\begin{matrix} M_1 = M_2^* \\ M_2 = M_1^* \end{matrix}} \right\} \begin{matrix} \text{both} \\ \text{real} \end{matrix}$$

if \textcircled{I} , $M_1, M_2 = 1 \Rightarrow M_1 = 1/M_1^*, M_2 = 1/M_2^* \Rightarrow \underline{M_1^2 = M_2^2 = 1}$
(trivial)

if (II) $\mu_1 \mu_2 = 1$; $\mu_1, \mu_2 \text{ real} \Rightarrow$

$$\Rightarrow x_1(t) = \mu^{t/T} \pi_1(t), \quad x_2(t) = \mu^{-t/T} \pi_2(t)$$

i.e. $\begin{cases} \uparrow \text{ increasing} \\ \downarrow \text{ decreasing} \end{cases}$ solution \Rightarrow parametric instability

[N.B. Exponential, not secular, growth]!

\Rightarrow "true" instability is possible

\rightarrow Some Calculation (as basic structure of the solution established).

Consider Mathieu's eqn:

$$\ddot{x} + \omega_0^2 [1 + h \cos[(2\omega_0 + \epsilon)t]] x = 0$$

bounds on ϵ for instability?

For solution, SHO \Rightarrow

$$x = a \cos(\omega_0 t) + b \sin(\omega_0 t)$$

so, in spirit of multiple-time-scale P.T.

(i.e. $\omega^2(t)$ enters via $h \ll 1 \Rightarrow$ expect slow time scale variation of coefficients)

$$x = a(t) \cos[(\omega_0 + \epsilon/2)t] + b(t) \sin[(\omega_0 + \epsilon/2)t]$$

\Downarrow coeffs become slowly varying

Plugging it in:

$$\ddot{x} = (a(t) \cos[(\omega_0 + \epsilon/2)t]) + O.T. \quad \text{other term}$$

$$= -(\omega_0 + \epsilon/2)^2 a(t) \cos[\] - 2(\omega_0 + \epsilon/2) \dot{a}(t) \sin[\] + \ddot{a} \cos[\] + O.T.$$

neglect \ddot{a}, \ddot{b} as h.o. in slowness (recall amplitude eqn. deriv.)

\Rightarrow ω_0^2 term, only

$$- (\omega_0 + \epsilon/2)^2 a(t) \cos[\] - 2\dot{a}(t) (\omega_0 + \epsilon/2) \sin[\]$$

$$- (\omega_0 + \epsilon/2)^2 b(t) \sin[\] + 2\dot{b}(t) (\omega_0 + \epsilon/2) \cos[\]$$

$$+ \omega_0^2 [a(t) \cos[\] + b(t) \sin[\]]$$

$$+ \omega_0^2 h \cos(2\omega_0 t) [a(t) \cos[\] + b(t) \sin[\]]$$

$$= 0$$

Now; - neglect $O(\epsilon^2)$ terms \Rightarrow only $\omega_0 \epsilon$ term survives.

- observe $\cos[(\omega_0 + \frac{\epsilon}{2})t] \cos[2\omega_0 t]$

$$= \frac{1}{2} \cos[3(\omega_0 + \epsilon/2)t] + \frac{1}{2} \cos[(\omega_0 + \epsilon/2)t]$$

Resonant contribution is interesting one, here

{ fast oscillation } \rightarrow Resonant with fundamental (i.e. expect h.o. in h)

⇒

$$\begin{aligned}
 & -\omega_0 \epsilon (a(t) \cos[\] + b(t) \sin[\]) \\
 & - 2 \dot{a} (\omega_0 + \epsilon/2) \sin[\] + 2 \dot{b} (\omega_0 + \epsilon/2) \cos[\] \\
 & + \frac{\omega_0^2 h}{2} [a(t) \cos[\] - b(t) \sin[\]] \\
 & = 0
 \end{aligned}$$

Regrouping coeffs. $\cos[\]$, $\sin[\]$;

$$\begin{aligned}
 & \sin[\] (-2\omega_0 \dot{a} - b\omega_0 \epsilon - \omega_0^2 h b/2) \\
 & + \cos[\] (2\dot{b}\omega_0 - a\epsilon\omega_0 + \frac{1}{2} h\omega_0^2 a) = 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow & (2\omega_0) \dot{a} + (\omega_0 \epsilon) b + (\frac{\omega_0^2 h}{2}) b = 0 \\
 & (2\omega_0) \dot{b} - (\omega_0 \epsilon) a + (\frac{\omega_0^2 h}{2}) a = 0
 \end{aligned}$$

⇒

$$\begin{cases}
 \dot{a} + (\epsilon/2) b + (\omega_0 h/4) b = 0 \\
 \dot{b} - (\epsilon/2) a + (\omega_0 h/4) a = 0
 \end{cases}$$

Basic
system
of
Eqs for
Amplitude
Variation

$$a(t) = a_0 e^{st}$$

$$b(t) = b_0 e^{st}$$

exponentially growing/damping solutions

⇒

$$5 a_0 + (\epsilon/2 + \omega_0 h/4) b_0 = 0$$

$$\left(-\frac{\epsilon}{2} + \frac{\omega_0 h}{4}\right) a_0 + 5 b_0 = 0$$

$$s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4} \left(\frac{\omega_0^2 h^2}{4} - \epsilon^2 \right)$$

⇒ Parametric instability criterion

Growth rate

Observe:

- instability for:

$$\epsilon^2 = (\gamma - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$$

$\omega_0 \rightarrow$ Fundamental
 $\gamma \rightarrow$ parametric variation freq.

amplitude of variation

$$h^2 > 4(\gamma - \omega_0)^2 / \omega_0^2$$

↑
 i.e. sufficiently close to resonance
 ⇒ growth.

for $(\gamma - 2\omega_0)^2 > \omega_0^2 h^2 / 4 \rightarrow$ oscillation

- amplitude of $\omega_0^2(t)$ variation sets proximity threshold

integer

more generally, can show when $n\gamma = 2\omega_0$
 \Rightarrow parametric resonance. of course, higher
 $n \Rightarrow$ resonance region $\sim h^n$

- with friction, find threshold for instability:

c.e. $(\gamma - 2\omega_0)^2 < \left[\left(\frac{1}{2} h \omega_0 \right)^2 - 4\alpha^2 \right]$

↑
friction coeff.

c.e. P.I. growth must be damped.
 Friction raises required h .

- Pumping on swing

\rightarrow "pumping" \rightarrow change of I

$$\ddot{\theta} + \frac{mgl\theta}{I(t)} = 0$$

$$I(t) = I_0 + \epsilon I_1(t)$$

$$\ddot{\theta} + \frac{g\theta}{l} + \frac{\epsilon g}{l} \frac{\Delta I(t)}{I} \theta = 0$$

$$+ \alpha \theta$$

need pump twice per cycle